Topology during Subdivision of Bézier Curves I: Angular Convergence & Homeomorphism

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Abstract

For Bézier curves, subdivision algorithms create control polygons as piecewise linear (PL) approximations that converge in terms of Hausdorff distance. We prove that the *exterior angles* of control polygons under subdivision converge to 0 at the rate of $O(\sqrt{\frac{1}{2^i}})$, where i is the number of subdivisions. This angular convergence is useful for determining topological features. We use it to show homeomorphism between a Bézier curve and its control polygon under subdivision. The constructive geometric proofs yield closed-form formulas to compute sufficient numbers of subdivision iterations to obtain small exterior angles and achieve homeomorphism.

Keywords: Bézier curve, subdivision, piecewise linear approximation, angular convergence, non-self-intersection, homeomorphism.

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1. Introduction

A Bézier curve (Definition 3.1) is characterized by an indexed set of points, which form a PL approximation of the curve, called a control polygon (Definition 3.1). The de Casteljau algorithm [8] is a subdivision algorithm associated to Bézier curves which recursively generate control polygons more closely approximating the curve under Hausdorff distance [21].

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There is contemporary interest [1, 2, 15, 19] to preserve topological characteristics such as homeomorphism and ambient isotopy between an initial geometric model and its approximation. In particular, the control polygon homeomorphic to a simple Bézier curve is also simple, so homeomorphism precludes undesired self-intersections, while the control polygon ambient isotopic to a Bézier curve has the same knot type as the Bézier curve. These are fundamental for applications in computer graphics, computer animation and scientific visualization.

However, there may be substantial topological differences between Bézier curves and their control polygons. First of all, Bézier curves and their control polygons are not necessarily homeomorphic. There are examples in the literature showing simple Bézier curves with self-intersecting control polygons or self-intersecting Bézier curves with simple control polygons [11, 23, 24]. Secondly, Bézier curves and their control polygons are not necessarily ambient isotopic. There is an example showing an unknotted Bézier curve with a knotted control polygon [3, 18]. An example of a knotted Bézier curve with an unknotted control polygon was constructed recently [11].

Control polygons converge under Hausdorff distance does not necessarily yield the equivalence relations mentioned above. We find that the angular convergence, which to the best of our knowledge has not been previously established, is useful for determining homeomorphism and knot type. Here, we first show the angular convergence and then prove the homeomorphism based on the angular convergence. We consider knot type separately and present that in the companion paper [12].

Computationally, it is known that the convergence in Hausdorff distance is exponential [22]. We show that the angular convergence rate is also exponential. Readers will find that the algorithmic efficiency of achieving the equivalence relations depends on these exponential convergence rates. Furthermore, we derive closed-form formulas to compute sufficient numbers of subdivision iterations to achieve the desired topological characteristics. These formulas rely upon the constructive geometric proofs presented here.

2. Related Work

The Angular Convergence we show here is based on a previous result showing that the discrete derivatives of the control polygons converge exponentially to the derivatives of the Bézier curve [20].

The topological characteristics for a Bézier curve and the control polygon have been studied in the literature. A previous proof [23] of homeomorphism under subdivision relied upon the hodograph[2] and did not provide the number of subdivision iterations. We provide a constructive geometric proof that sufficiently many subdivisions will produce a control polygon homeomorphic to a given Bézier curve, and a sufficient number of subdivision iterations is also established. Topologically reliable approximation in terms of homeomorphism of composite Bézier curves was also studied by the paper [6], which used algorithmic techniques that do not completely rely upon de Casteljau algorithm, techniques with choosing so called "significant points" first.

Ambient isotopy is a stronger notion of equivalence than homeomorphism. There exist a result [9] establishing this equivalence for geometric models, by a certain algorithm distinct from subdivision. Besides, recent papers [4, 13] present algorithms to compute the isotopic polygonal approximation for $2D$ algebraic curves. Computational techniques for establishing isotopy and homotopy have been established regarding algorithms for point-cloud data by "distance-like functions" [5]. In the companion paper [12], the input data there is different of a Bézier curve, with the important additional result of the number of subdivision iterations needed to attain an ambient isotopic approximation of the original Bézier curve.

Specifically for $3D$ Bézier curves under subdivision, the paper [19] showed ambient isotopy only for curves of low degree (less than 4), where a crucial unknotting condition was trivial. Deriving a comparable unknotting condition for higher degrees entailed significant new arguments that depend on our angular convergence.

A geometric technique we use is constructing a tubular neighborhood for a Bézier curve, whose boundary is called a pipe surface. Pipe surfaces have been studied since the 19th century [17], but the presentation here follows a contemporary source [14]. These contemporary authors [14] provide Bézier curves considered here. These authors perform a thorough analysis and description of the end conditions of open curves. The junction points of a Bézier curve are merely a special case of that analysis.

[2]The derivative of a Bézier curve is also expressed as a Bézier curve, known as the *hodograph* [8].

3. Definitions and Notation

Mathematical definitions, notation and a fundamental supportive theorem are presented in this section. More specialized definitions will follow in appropriate sections. The standard Euclidean norm will be denoted by $|| \quad ||$.

Definition 3.1. *The parameterized* ***Bézier curve****, denoted as $\mathcal{B}(t)$, of degree n with control points $P_m \in \mathbb{R}^3$ is defined by*

$$\mathcal{B}(t) = \sum_{m=0}^{n} B_{m,n}(t) P_m, t \in [0,1],$$

where $B_{m,n}(t) = \binom{n}{m} t^m (1-t)^{n-m}$ and the PL curve given by the points $\{P_0, P_1, \ldots, P_n\}$ is called its ***control polygon****. When $P_0 = P_n$, the control polygon is closed. Otherwise when $P_0 \neq P_n$, it is open.*

In order to avoid technical considerations and to simplify the exposition, the class of Bézier curves considered will be restricted to those where the first derivative never vanishes.

Definition 3.2. *A differentiable curve is said to be* ***regular*** *if its first derivative never vanishes.*

Definition 3.3. *A curve is said to be* ***simple*** *if it is non-self-intersecting.*

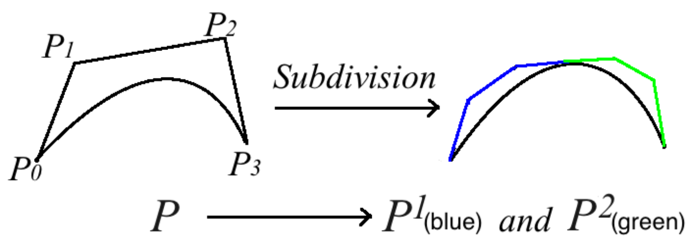
The Bézier curve of Definition 3.1 is typically called a *single segment Bézier curve*, while a *composite Bézier curve* is created by joining two or more single segment Bézier curves at their common end points.

We use $\mathcal{B}$ to denote a simple, regular, C^1, composite Bézier curve in $\mathbb{R}^3$, throughout the paper.

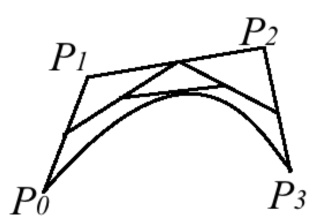
Definition 3.4. [21] *Let X and Y be two non-empty subsets of a metric space (M, d). We define their* ***Hausdorff distance*** *$\mu(X, Y)$ by*

$$\mu(X,Y) := \max\{\sup_{x \in X} \inf_{y \in Y} d(x,y), \sup_{y \in Y} \inf_{x \in X} d(x,y)\}.$$

Subdivision algorithms are fundamental for Bézier curves [8] and a brief overview is given here. Figure 1 shows the first step of the de Casteljau algorithm with an input value of $\frac{1}{2}$ on a single Bézier curve. For ease of exposition, the de Casteljau algorithm with this value of $\frac{1}{2}$ is assumed, but other fractional values can be used with appropriate minor modifications to the analyses presented. The initial control polygon P is used as input to generate local PL approximations, P^1 and P^2, as Figure 1(a) shows. Their union, $P^1 \cup P^2$, is then a new PL curve whose Hausdorff distance is closer to the curve than that of P.



(a) Initial and resultant curves



(b) Subdivision process

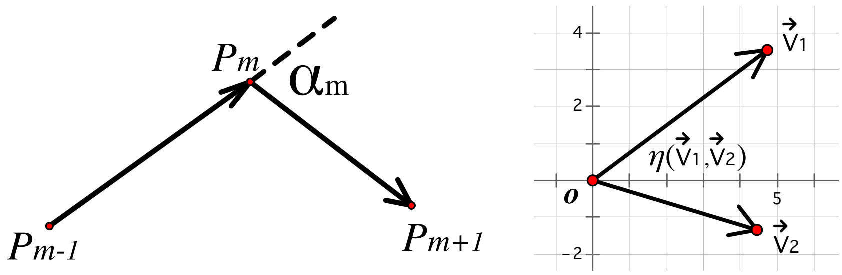
Figure 1: A subdivision with parameter $\frac{1}{2}$

A summary is that subdivision proceeds by selecting the midpoint of each segment of P and these midpoints are connected to create new segments, as Figure 1(b) shows. Recursive creation and connection of midpoints continues until a final midpoint is selected. The union of the segments from the last step then forms a new PL curve. Termination is guaranteed since P has only finitely many segments.

After i iterations, the subdivision process generates 2^i PL sub-curves, each being a control polygon for part of the original curve [8], which will be

referred to as a **sub-control polygon**[3], denoted by P^k for $k = 1, 2, 3, \ldots, 2^i$. Each P^k has n points and their union $\bigcup_k P^k$ forms a new PL curve that converges in Hausdorff distance to approximate the original Bézier curve. The Bézier curve defined by $\bigcup_k P^k$ is exactly the same Bézier curve defined by the original control points $\{P_0, P_1, \ldots, P_n\}$ [10]. So $\bigcup_k P^k$ is a new control polygon of the Bézier curve.

Exterior angles were defined [16] in the context of closed PL curves, but are adapted here for both closed and open PL curves. Exterior angles unify the concept of total curvature for curves that are PL or differentiable.



(a) Orientated lines

(b) Vectors

Figure 2: An exterior angle

Definition 3.5. *The* ***exterior angle*** *between two oriented line segments, denoted as* $\overrightarrow{P_{m-1}P_m}$ *and* $\overrightarrow{P_mP_{m+1}}$*, is the angle between the extension of* $\overrightarrow{P_{m-1}P_m}$ *and* $\overrightarrow{P_mP_{m+1}}$*, as shown in Figure 2(a). Let the measure of the exterior angle to be* α_m *satisfying:*

$$0 \leq \alpha_m \leq \pi.$$

This definition naturally generalizes to any two vectors, $\vec{v}_1$ *and* $\vec{v}_2$*, by joining these vectors at their initial points, while denoting the measure between them as* $\eta(\vec{v}_1, \vec{v}_2)$*, as indicated in Figure 2(b).*

Definition 3.6. *The* ***curvature*** *of a* C^2 *curve* $C(t)$ *parametrized on* $[a, b]$ *is given by*

$$\kappa(t) = \frac{||C'(t) \times C''(t)||}{||C'(t)||^3}, \quad t \in [a, b]. \tag{1}$$

[3]Note that by the subdivision process, each sub-control polygon of a simple Bézier curve is open.

Its ***total curvature*** [7] *is the integral:* $\int_a^b |\kappa(t)|\, dt$.

Total curvature can be defined for both C^2 and PL curves. In both cases, the total curvature is denoted by $T_\kappa(\cdot)$. The unified terminology is invoked in Fenchel's theorem, which follows and is fundamental to the work presented here.

Definition 3.7. [16] *The* ***total curvature*** *of a PL curve in $\mathbb{R}^3$ is the sum of the measures of the exterior angles.*

Fenchel's Theorem [7] presented below is applicable both to PL curves and to differentiable curves.

Theorem 3.1. [7, Fenchel's Theorem] *The total curvature of any closed curve is at least 2π, with equality holding if and only if the curve is convex.*

Denote a PL curve with vertices $\{P_0, P_1, \cdots, P_n\}$ by P, and the uniform parametrization [20] of P over $[0,1]$ by $l(P)_{[0,1]}$. That is:

$$l(P)_{[0,1]}(\frac{j}{n}) = P_j \ for\ j = 0, 1, \cdots, n$$

and $l(P)_{[0,1]}$ interpolates linearly between vertices.

Definition 3.8. ***Discrete derivatives*** *[20] are first defined at the parameters $t_j = \frac{j}{n}$, where*

$$l(P)_{[0,1]}(t_j) = P_j$$

for $j = 0, 1, \cdots, n-1$. Let

$$P'_j = l'(P)_{[0,1]}(t_j) = \frac{P_{j+1} - P_j}{t_{j+1} - t_j}.$$

Denote $P' = (P'_0, P'_1, \cdots, P'_{n-1})$. Then define the discrete derivative for $l(P)_{[0,1]}$ as:

$$l'(P)_{[0,1]} = l(P')_{[0,1]}.$$

Intuitively, the first discrete derivatives are similar to the slopes defined for univariate real-valued functions within an introductory calculus course.

4. Angular Convergence under Subdivision

Using the notation introduced in Section 3, for a sub-control polygon P^k, we consider the following analysis, where, for simplicity of notation, we repress the superscript and denote this arbitrary sub-control polygon simply as P, where P has the corresponding parameters of the indicated control points by $t_0, t_1, \cdots, t_n$. And let $l(P, i)$ be the uniform parameterization [20] of P on $[\frac{k-1}{2^i}, \frac{k}{2^i}]$ $k \in \{1, 2, 3, \ldots, 2^i\}$. That is

$$l(P, i) = l(P)_{[\frac{k-1}{2^i}, \frac{k}{2^i}]} \quad and \quad l(P, i)(t_m) = P_m \; for \;\; m = \{0, 1, \cdots, n\},$$

where $t_m = \frac{k-1}{2^i} + \frac{m}{n2^i}$. Note from the domain of $l(P, i)$ that

$$t_n - t_0 = \frac{1}{2^i} \;\; and \;\; t_m - t_{m-1} = \frac{1}{n2^i} \;\; for \;\; m = \{1, \cdots, n\}. \qquad (2)$$

Furthermore, let

$$\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$$

be the corresponding measures of exterior angles of P (Definition 3.5). Treat the discrete derivative $l'(P, i)(t)$ at some $t \in [0, 1]$ as a point in $\mathbb{R}^3$, then the Euclidian distance between two such discrete derivatives refers to the Euclidian distance between two points. Consider the Euclidian distance between the discrete derivatives of the two consecutive segments, that is $||l'(P, i)(t_m) - l'(P, i)(t_{m-1})||$. We will show a rate of $O(\frac{1}{2^i})$ for the convergence

$$||l'(P, i)(t_m) - l'(P, i)(t_{m-1})|| \to 0 \text{ as } i \to \infty.$$

This will imply that $\cos(\alpha_m) \to 1$ with the same rate and that $\alpha_m \to 0$ at a rate of $O(\sqrt{\frac{1}{2^i}})$.

Lemma 4.1. *For a C^1, composite Bézier curve $\mathcal{B}$, the value $||l'(P, i)(t_m) - l'(P, i)(t_{m-1})||$ converges uniformly for all m and k to zero at a rate of $O(\frac{1}{2^i})$.*

Proof:

$$||l'(P, i)(t_m) - l'(P, i)(t_{m-1})||$$

$$\leq ||l'(P, i)(t_m) - \mathcal{B}'(t_m)|| + ||\mathcal{B}'(t_m) - \mathcal{B}'(t_{m-1})|| + ||\mathcal{B}'(t_{m-1}) - l'(P, i)(t_{m-1})||. \qquad (3)$$

The first and the third terms converge to 0 at a rate of $O(\frac{1}{2^i})$, because the discrete derivative converges to the derivative of the original curve with this rate [20].

For ease of exposition in this proof, we will initially assume that $\mathcal{B}$ is C^2, but then complete the proof by extending this special case to the more general assumption that $\mathcal{B}$ is C^1. If $\mathcal{B}$ is C^2, then for the second term, the Mean Value Theorem on $(0,1)$ provides that there exists some $s \in (t_m, t_{m-1})$ such that

$$||\mathcal{B}'(t_m) - \mathcal{B}'(t_{m-1})|| = ||\mathcal{B}''(s)|| \cdot |t_m - t_{m-1}|. \tag{4}$$

Let $\gamma = \sup_{[0,1]} ||\mathcal{B}''(t)||$, which exists because of the continuity of $\mathcal{B}''$ over $[0,1]$. It follows that

$$||\mathcal{B}'(t_m) - \mathcal{B}'(t_{m-1})|| \leq \sup_{[0,1]} ||\mathcal{B}''(t)|| \cdot |t_m - t_{m-1}| = \frac{\gamma}{n2^i}. \tag{5}$$

The second equality holds by Equation (2). So $||l'(P,i)(t_m) - l'(P,i)(t_{m-1})||$ converges to zero at a rate of $O(\frac{1}{2^i})$.

The curve $\mathcal{B}$ is only assumed to be C^1 at its junction points, so it is natural to question the validity of Equation 4 and Inequality 5 in the C^1. Note that Equation 4 is over the *open* interval $(0,1)$, so no difficulties occur relative to the junction points. However, more subtle consideration is necessary for Inequality 5. Each segment of the composite Bézier curve is a polynomial defined over $[0,1]$, so the definition of γ as a supremum over $[0,1]$ still exists.

Theorem 4.1 (Angular Convergence). *For a C^1, composite Bézier curve $\mathcal{B}$, the exterior angles of the PL curves generated by subdivision converge uniformly to 0 at a rate of $O(\sqrt{\frac{1}{2^i}})$.*

Proof: Since $\mathcal{B}(t)$ is assumed to be regular and C^1, the non-zero minimum of $||\mathcal{B}'(t)||$ over the compact set [0,1] is obtained. For brevity, the notations of $u_i = l'(P,i)(t_m)$, $v_i = l'(P,i)(t_{m-1})$ and $\alpha = \alpha_m$ (measure of an arbitrary exterior angle) are introduced. The convergence of u_i to $\mathcal{B}'(t_m)$ [20] implies that $||u_i||$ has a positive lower bound for i sufficiently large, denoted by λ.

Lemma 4.1 gives that $||u_i - v_i|| \to 0$ as $i \to \infty$ at a rate of $O(\frac{1}{2^i})$. This implies: $||u_i|| - ||v_i|| \to 0$ as $i \to \infty$ at a rate of $O(\frac{1}{2^i})$.

Consider the following where each multiplication between vectors is the dot product:

$$1 - \cos(\alpha) = 1 - \frac{u_i v_i}{||u_i|| \cdot ||v_i||}$$

$$= \frac{||u_i|| \cdot ||v_i|| - v_i v_i + v_i v_i - u_i v_i}{||u_i|| \cdot ||v_i||}$$

$$\leq \frac{||u_i|| - ||v_i||}{||u_i||} + \frac{||v_i - u_i||}{||u_i||} \leq \frac{||u_i|| - ||v_i||}{\lambda} + \frac{||v_i - u_i||}{\lambda} \leq \frac{2||v_i - u_i||}{\lambda} \quad (6)$$

It follows from Lemma 4.1 that the right hand side converges to 0 at a rate of $O(\frac{1}{2^i})$. Consequently by the above inequality $1 - \cos(\alpha) \to 0$ with the same rate. It follows from the continuity of $arc \cos$ that α converges to 0 as $i \to \infty$.

To obtain the convergence rate, taking the power series expansion of cos we get

$$\begin{aligned} 1 - \cos(\alpha) &\geq (\alpha)^2 \cdot (\frac{1}{2} - |\frac{(\alpha)^2}{4!} - \frac{(\alpha)^4}{6!} + \cdots |) \\ &= (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot |\frac{1}{4!} - \frac{(\alpha)^2}{6!} + \cdots |) \end{aligned} \quad (7)$$

Note that for $1 > \alpha$,

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots > |\frac{1}{4!} - \frac{(\alpha)^2}{6!} + \cdots |. \quad (8)$$

Combining Inequality 7 and 8 we have,

$$1 - \cos(\alpha) > (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot e).$$

For any $0 < \tau < \frac{1}{2}$, sufficiently many subdivisions will guarantee that α is small enough such that $1 > \alpha$ and $\tau > (\alpha)^2 \cdot e$. Thus

$$1 - \cos(\alpha) > (\alpha)^2 \cdot (\frac{1}{2} - \alpha^2 \cdot e) > (\alpha)^2 \cdot (\frac{1}{2} - \tau) > 0.$$

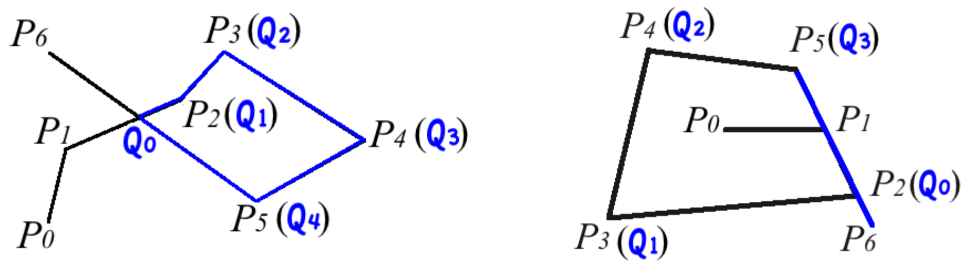
Then convergence of the left hand side implies that α converges to 0 at a rate of $O(\sqrt{\frac{1}{2^i}})$.

5. Topological Equivalence (Homeomorphism)

The terminology of *topological equivalence* is classical, referring to two sets being homeomorphic. We present sufficient conditions for a homeomorphism between a subdivided control polygon and its associated Bézier curve. To establish a homeomorphism under subdivision, we first establish a local homeomorphism between a sub-control polygon and the corresponding sub-curve of $\mathcal{B}$, and then establish a global homeomorphism between the control polygon and $\mathcal{B}$.

5.1. Local arguments for topological equivalence

A similar lemma has appeared [23] without proof. It is likely that the proof is well known, possibly as a 'folk theorem', but we are unable to find a reference in the literature, so its proof is provided here, for the sake of completeness.



(a) A single point intersection (b) Coincident line segments

Figure 3: Self-intersecting PL curves in 3D

Lemma 5.1 (Non-self-intersection criteria[4]). *Let* $P = (P_0, P_1, \cdots, P_n)$ *be an open* PL *curve in* $\mathbb{R}^3$*. If* $T_\kappa(P) = \sum_{j=1}^{n-1} \alpha_j < \pi$*, then* P *is simple.*

Proof: Assume to the contrary that P is self-intersecting. Then there must exist at least one closed loop. (The assumption $\sum_{j=1}^{n-1} \alpha_j < \pi$ precludes the case of two consecutive edges being coincident.) Arbitrarily choose one such loop. There are two cases to consider: a single point intersection as in Figure 3(a) and the coincident case as in Figure 3(b).

Case 1 (A single point intersection): Label the single intersection as Q_0 to be the first vertex of the loop (Figure 3(a)). Label the other vertices following the orientation of P and denote this loop by

$$\bar{Q} = (Q_0, Q_1, \cdots, Q_{n'}, Q_0),$$

for an appropriately chosen value of n'. Denote the measure of the corresponding exterior angle of $\bar{Q}$ at $Q_0, Q_1, \cdots, Q_{n'}$ by $\beta_0, \beta_1, \cdots, \beta_{n'}$. Note that $\beta_0 \leq \pi$ (Definition 3.5). Note also that each exterior angle β_l (for $l = 1, 2, \cdots, n'$) of $\bar{Q}$ (except the initial angle β_0) is an exterior angle of P. So

$$\sum_{l=0}^{n'} \beta_l = \beta_0 + \sum_{l=1}^{n'} \beta_l \leq \pi + \sum_{j=1}^{n-1} \alpha_j.$$

Since $\sum_{j=1}^{n-1} \alpha_j < \pi$ we get

$$\sum_{l=0}^{n'} \beta_l \leq \pi + \sum_{j=1}^{n-1} \alpha_j < 2\pi.$$

But for the closed loop $\bar{Q}$, $\sum_{l=0}^{n'} \beta_l \geq 2\pi$ (Theorem 3.1), which is a contradiction.

Case 2 (Coincident line segments): The loop is formed by a subset of the vertices $\{P_0, P_1, \cdots, P_n\}$ in this case (Figure 3(b)). Pick an arbitrary point from the subset and relabel it as Q_0. Label the other vertices following the orientation of P and use the notation $\bar{Q} = (Q_0, Q_1, \cdots, Q_{n'}, Q_0)$ and $\beta_0, \beta_1, \cdots, \beta_{n'}$ as the above. Note for this case that, each exterior angle β_l of $\bar{Q}$ is an exterior angle of P. So

$$\sum_{l=0}^{n'} \beta_l \leq \sum_{j=1}^{n-1} \alpha_j.$$

And $\sum_{j=1}^{n-1} \alpha_j < \pi$, so $\sum_{l=0}^{n'} \beta_l < \pi$. But for the closed loop $\bar{Q}$, $\sum_{l=0}^{n'} \beta_l \geq 2\pi$ (Theorem 3.1), which is a contradiction. □

Theorem 5.1 (Simple sub-control polygons). *For a C^1, composite Bézier curve $\mathcal{B}$, there exists a sufficiently large value of i, such that after i-many subdivisions, each of the sub-control polygons generated as P^k for $k = 1, 2, 3, \ldots, 2^i$ will be simple.*

Proof: For each P^k, the measures of the exterior angles of P^k converge uniformly to zero as i increases (Theorem 4.1). Each open P^k has n edges. Denote the $n-1$ exterior angles of each P^k by α_j^k, for $j = 1, \ldots, n-1$ and for $k = 1, 2, 3, \ldots, 2^i$. Then there exists i sufficiently large such that

$$\sum_{j=1}^{n-1} \alpha_j^k < \pi,$$

for each $k = 1, 2, 3, \ldots, 2^i$. Use of Lemma 5.1 completes the proof. □

5.2. Global arguments for topological equivalence

The proof techniques for homeomorphism rely upon the sub-control polygons to be pairwise disjoint, except at their common end points. Denote two generated sub-control polygons of $\mathcal{B}$ as

$$P = (P_0, P_1, \cdots, P_n) \text{ and } Q = (Q_0, Q_1, \cdots, Q_n).$$

Definition 5.1. *The sub-control polygons P and Q are said to be **consecutive** if the last vertex P_n of P is the first vertex Q_0 of Q, that is, $P_n = Q_0$.*

Remark 5.1. *For $\mathcal{B}$, the C^1 assumption ensures that the segments $\overrightarrow{P_{n-1}P_n}$ and $\overrightarrow{Q_0Q_1}$ are collinear, so the exterior angle at the common point $P_n = Q_0$ is either 0 or π. But the regularity assumption ensures that the exterior angle can not be π, so that the exterior angle at the common point is 0.*

Lemma 5.2 extends to arbitrary degree Bézier curves from a previously established result that was restricted to cubic Bézier curves [25], as used in the proof of isotopy under subdivision for low-degree Bézier curves [19].

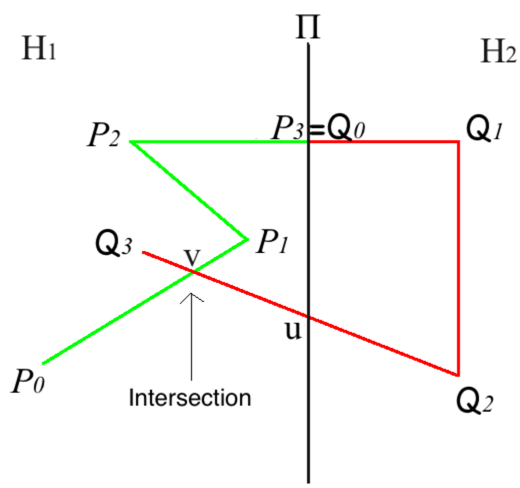


Figure 4: Intersecting consecutive sub-control polygons

Lemma 5.2. *Let Π be the plane normal to a sub-control polygon at its initial vertex. If the total curvature of the sub-control polygon is less than $\frac{\pi}{2}$, then the initial vertex is the only single point where the plane intersects the sub-control polygon.*

Proof: Denote the sub-control polygon as $Q = (Q_0, Q_1, \ldots, Q_n)$, where Figure 4 shows an orthogonal projection of this 3D geometry. Assume to the contrary that $\Pi \cap Q$ contains a point u where $u \neq Q_0$. Assume without loss of generality that a segment $Q_{m-1}Q_m$ intersects Π at the point u for some $m \in \{2, 3, \cdots, n\}$. This would produce a closed PL curve denoted as

$$\bar{J} = (Q_0, Q_1, \cdots, Q_{m-1}, u, Q_0).$$

Denote the exterior angle of $\bar{J}$ at Q_0 as θ and note that $\theta = \frac{\pi}{2}$.Denote the exterior angle of $\bar{J}$ at u as γ. Then $\gamma \leq \pi$ (Definition 3.5). Denote the other exterior angles of $\bar{J}$ as $\beta_i, i = 1, \ldots, m-1$. By Theorem 3.1 we have

$$\theta + \sum_{i=1}^{m-1} \beta_i + \gamma \geq 2\pi.$$

Thus

$$\sum_{i=1}^{m-1} \beta_i \geq 2\pi - \theta - \gamma \geq \frac{\pi}{2}.$$

These β_i are also exterior angles of Q resulting in lower bound for the total curvature of Q as given by $T_\kappa(Q) \geq \sum_i^{m-1} \beta_i \geq \pi/2$, which is a contradiction to $T_\kappa(Q) < \frac{\pi}{2}$. □

Lemma 5.3. *Recall that $\mathcal{B}$ denotes a simple, regular, C^1, composite Bézier curve in $\mathbb{R}^3$. Let w be a point of $\mathcal{B}$ where $\mathcal{B}$ is subdivided and let Π be the plane normal to $\mathcal{B}$ at w. Then there exists a subdivision of $\mathcal{B}$ such that the sub-control polygon ending at w and the sub-control polygon beginning at w intersect Π only at the single point w.*

Proof: The plane Π separates $\mathbb{R}^3$ into two disjoint open half-spaces, denoted as H_1 and H_2, such that $\mathbb{R}^3 = H_1 \cup \Pi \cup H_2$ and $H_1 \cap H_2 = \emptyset$. By Remark 5.1, the exterior angle at $\{w\}$ is 0.

Perform sufficient many subdivisions so that the control polygon ending at w, denoted by P , and the control polygon beginning at w, denoted by Q, each have total curvature less than $\frac{\pi}{2}$ by Theorem 4.1. Therefore, by Lemma 5.2 the only point where P or Q intersect Π is at w. □

This global homeomorphism will be proven by reliance upon pipe surfaces, which are defined below.

Definition 5.2. *The* ***pipe surface*** *of radius r of a parameterized curve $\boldsymbol{c}(t)$, where $t \in [0,1]$ is given by*

$$\boldsymbol{p}(t,\theta) = \ \boldsymbol{c}(t) + r[cos(\theta)\ \boldsymbol{n}(t) + sin(\theta)\ \boldsymbol{b}(t)],$$

where $\theta \in [0, 2\pi]$ and $\boldsymbol{n}(t)$ and $\boldsymbol{b}(t)$ are, respectively, the normal and binormal vectors at the point $\mathbf{c}(t)$, as given by the Frenet-Serret trihedron. The curve $\mathbf{c}$ is called a spine curve.

For $\mathcal{B}$ and i subdivisions, with resulting sub-control polygons P^k for $k = 1, \ldots, 2^i$, let $S_{\mathcal{B}}(r)$ be a pipe surface of radius r for $\mathcal{B}$ so that $S_r(\mathcal{B})$ is nonsingular. For each $k = 1, \ldots, 2^i$, denote

- the parameter of the initial point of P^k by t_0^k, and that of the terminal point by t_n^k
- the normal disc of radius r centered at $\mathcal{B}(t)$ as $D_r(t)$,
- the union $\bigcup_{t\in[t_0^k, t_n^k]} D_r(t)$ by Γ_k, and designate it as a **pipe section**.

Theorem 5.2 (A simple homeomorphic control polygon). *Sufficient subdivisions will yield a simple control polygon that is homeomorphic to $\mathcal{B}$.*

Proof: By Theorem 4.1, we can take ι_1 subdivisions so that $T_\kappa(P^k) < \pi/2$, for each sub-control polygon P^k. By Lemma 5.1, this choice of ι_1 guarantees that each P^k is simple. By the convergence in Hausdorff distance under subdivision [22], we can take ι_2 subdivisions such that the control polygon generated by ι_2 subdivision fits inside the pipe surface $S_r(\mathcal{B})$. Choose $\iota = \max\{\iota_1, \iota_2\}$. By Lemma 5.3, this choice of ι ensures that each P^k fits inside the corresponding Γ_k. This plus the fact that P^k is simple shows that the control polygon, $\bigcup_{k=1}^{2^i} P^k$, is simple, which implies the homeomorphism. □

6. Sufficient Subdivision Iterations

In this section, we shall establish: (1) numbers of subdivisions for small exterior angles based on Angular Convergence; and (2) numbers of subdivisions for homeomorphism.

From the previous sections we know that the homeomorphism is obtained by subdivision based on two criteria: (1) Angular Convergence; and (2) convergence in distance. So the speed of achieving this topological characteristics

is determined by the Angular Convergence rate and the convergence rate in distance which are both exponential. Here, we further find closed-form formulas to compute sufficient numbers of subdivision iterations to achieve these properties.

Definition 6.1. *Let P denote a control polygon of a Bézier curve, and let P_x denote an ordered list of all of x-coordinates of P (with similar meaning given to P_y for the y-coordinates and to P_z for the z-coordinates). Let*

$$\| \triangle_2 P_x \|_\infty = \max_{0<m<n} |P_{m-1,x} - 2P_{m,x} + P_{m+1,x}|$$

be the maximum absolute second difference of the x-coordinates of control points, (with similar meanings for the y and z coordinates) . Let

$$\Delta_2 P = (\| \triangle_2 P_x \|_\infty, \| \triangle_2 P_y \|_\infty, \| \triangle_2 P_z \|_\infty),$$

(i.e.) a vector with 3 values.

Definition 6.2. *The* ***parameter measure distance***[5] [22] *between a Bézier curve $\mathcal{B}$ and the control polygon $l(P,i)$ generated by i subdivisions is given by*

$$\max_{t\in[0,1]} ||l(P,i)(t) - \mathcal{B}(t)||.$$

Lemma 6.1. *The parameter measure distance between the Bézier curve and its control polygon after ith-round subdivision is bounded by*

$$\frac{1}{2^{2i}} N_\infty(n) ||\Delta_2 P||, \tag{9}$$

where

$$N_\infty(n) = \frac{\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil}{2n}.$$

Proof: A published lemma [22, Lemma 6.2] proves a similar result restricted to scalar valued polynomials. We consider coordinate-wise and apply this result to the x, y, and z coordinates respectively, so that the distance of

[5] The paper [22] uses 'distance', but we adopt this terminology 'parameter measure distance' instead.

the x-coordinates of the Bézier curve and its control polygon after ith-round subdivision is bounded by

$$\frac{1}{2^{2i}} N_\infty(n) \parallel \Delta_2 P_x \parallel_\infty,$$

with similar expressions for the y and z coordinates. Taking the Euclidean norm of the indicated three x, y and z bounds yields the upper bound given by (9), an upper bound of the parameter measure distance between the Bézier curve and its control polygon after the ith subdivision. □

For convenience, denote the above bound in distance as:

$$B_{dist}(i) := \frac{1}{2^{2i}} N_\infty(n) ||\Delta_2 P||. \tag{10}$$

Lemma 6.2. *After i subdivision iterations, the distance between $l'(P, i)$ and $\mathcal{B}'$ is bounded by $B'_{dist}(i)$, where*

$$B'_{dist}(i) := \frac{1}{2^{2i}} N_\infty(n-1) ||\Delta_2 P'||, \tag{11}$$

and P' that consists of $n-1$ control points is the control polygon of $\mathcal{B}'$.

Proof: The derivative $l'(P, i)$ of the control polygon for a Bézier curve is identical to the control polygon $l(P', i)$ for the Bézier curve's first derivative by [20, Lemma 6]. That is,

$$l'(P, i) = l(P', i).$$

Since (Lemma 6.1)

$$\max_{t \in [0,1]} ||l(P', i)(t) - \mathcal{B}'(t)|| \leq B'_{dist}(i),$$

we have

$$\max_{t \in [0,1]} ||l'(P, i)(t) - \mathcal{B}'(t)|| \leq B'_{dist}(i). \tag{12}$$

□

6.1. Sufficient subdivision iterations for small exterior angles

Assume ν is a small measure of angle between 0 and π. We shall find how many subdivisions will generate a control polygon such that the measure α of each exterior angle satisfies

$$\alpha < \nu. \tag{13}$$

Recall the proof of Angular Convergence (Theorem 4.1). Consider two arbitrary consecutive derivatives $u_i = l'(P,i)(t_m)$ and $v_i = l'(P,i)(t_{m-1})$ and the corresponding exterior angle α. Recall that in Section 4 we had Inequality 6:

$$1 - \cos(\alpha) \leq \frac{2||v_i - u_i||}{||u_i||}.$$

Recall the proof of Lemma 4.1 where Inequality 3 is:

$$||u_i - v_i||$$

$$\leq ||l'(P,i)(t_m) - \mathcal{B}'(t_m)|| + ||\mathcal{B}'(t_m) - \mathcal{B}'(t_{m-1})|| + ||\mathcal{B}'(t_{m-1}) - l'(P,i)(t_{m-1})||,$$

and Inequality 5 is

$$||\mathcal{B}'(t_m) - \mathcal{B}'(t_{m-1})|| \leq \sup_{[0,1]} ||\mathcal{B}''(t)|| \cdot |t_m - t_{m-1}| = \frac{\gamma}{n2^i},$$

where $\gamma = \sup_{[0,1]} ||\mathcal{B}''(t)||$. Combining the above inequalities yields

$$1 - \cos(\alpha) \leq \frac{2(||l'(P,i)(t_m) - \mathcal{B}'(t_m)|| + ||\mathcal{B}'(t_{m-1}) - l'(P,i)(t_{m-1})|| + \gamma/(n2^i))}{||u_i||}.$$

Using Lemma 6.2 we get

$$1 - \cos(\alpha) \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{||u_i||}. \tag{14}$$

Let $\sigma = \min\{||\mathcal{B}'(t)|| : t \in [0,1]\}$. The regularity of $\mathcal{B}$ ensures that $\sigma > 0$ and the continuity of $\mathcal{B}'$ on the compact interval $[0,1]$ ensures that the minimum exists. Recall $u_i = l'(P,i)(t_m)$ for some $t_m \in [0,1]$. So it follows from Inequality 12 that

$$||\mathcal{B}'(t_m)|| - ||u_i|| \leq B'_{dist}(i).$$

Solving the inequality we get

$$||u_i|| \geq ||\mathcal{B}'(t_m)|| - B'_{dist}(i) \geq \sigma - B'_{dist}(i).$$

In order to have $u_i \neq 0$, it is sufficient to perform enough subdivisions such that

$$||u_i|| \geq \sigma - B'_{dist}(i) > 0,$$

that is $B'_{dist}(i) < \sigma$. By the definition (Equation 11) of $B'_{dist}(i)$ we set,

$$\frac{1}{2^{2i}} N_\infty(n-1) \parallel \triangle_2 P' \parallel < \sigma.$$

Therefore for $B'_{dist}(i) < \sigma$, it suffices to have[6]

$$i > \frac{1}{2}\log\left(\frac{N_\infty(n-1) \parallel \triangle_2 P' \parallel}{\sigma}\right) = N_1. \tag{15}$$

It is worth noting that N_1 is a subdivision number depending on variables P', n and σ.

After the i subdivision iterations, whenever $i > N_1$, then $B'_{dist}(i) < B'_{dist}(N_1)$, because $B'_{dist}(i)$ is a strictly decreasing function (Equation 11). So it follows from Inequality 14 that whenever $i > N_1$,

$$1 - \cos(\alpha) \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{|\sigma - B'_{dist}(i)|} \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{\sigma - B'_{dist}(N_1)}.$$

To obtain $\alpha < \nu$ (Inequality 13), it suffices to have that $1 - cos(\alpha) < 1 - cos(\nu)$. Now choose i large enough so that

$$1 - \cos(\alpha) \leq \frac{2(2B'_{dist}(i) + \gamma/(n2^i))}{\sigma - B'_{dist}(N_1)} < 1 - \cos(\nu). \tag{16}$$

The second inequality of Inequality 16 implies that

$$2B'_{dist}(i) + \frac{\gamma}{n2^i} < \frac{1}{2}(1 - \cos(\nu))(\sigma - B'_{dist}(N_1)).$$

We could solve the above inequality for i, but we can avoid this complicated computation by noting that $2B'_{dist}(i)$ is much smaller than $\frac{\gamma}{n2^i}$ when

[6] Throughout this paper, we use log for $\log_2$.

i is large. (Note that $\gamma = \sup_{[0,1]} ||\mathcal{B}''(t)||$. If $\gamma = 0$, then the Bézier curve would be a straight line segment. We exclude this trivial case and assume that $\gamma > 0$.)

So let $2B'_{dist}(i) < \frac{\gamma}{n2^i}$, that is

$$i > \log(\frac{2nN_\infty(n-1) \parallel \triangle_2 P' \parallel}{\gamma}) = N_2. \tag{17}$$

So for $i > N_2$ we get

$$2B'_{dist}(i) + \frac{\gamma}{n2^i} < 2\frac{\gamma}{n2^i}.$$

Then we consider

$$2\frac{\gamma}{n2^i} < \frac{1}{2}(1-\cos(\nu))(\sigma - B'_{dist}(N_1)). \tag{18}$$

Solve Inequality 18 and get

$$\frac{1}{2^i} < \frac{n}{4\gamma}(1-\cos(\nu))(\sigma - B'_{dist}(N_1)).$$

So

$$i > \log(\frac{4\gamma}{n(1-\cos(\nu))(\sigma - B'_{dist}(N_1))}).$$

To simplify this expression, let

$$c_{\mathcal{B}} = \frac{4\gamma}{n(\sigma - B'_{dist}(N_1))},$$

which is a constant for a given $\mathcal{B}$, and let

$$f(\nu) = \frac{c_{\mathcal{B}}}{1-\cos\nu}. \tag{19}$$

Then, we have

$$i > \log(f(\nu)).$$

Theorem 6.1. *Given any $\nu > 0$, there exists an integer $N(\nu)$ defined by*

$$N(\nu) = \max\{N_1, N_2, \log(f(\nu))\} \tag{20}$$

where N_1, N_2 and $f(\nu)$ are given by Equations 15 17 and 19 respectively, such that each exterior angle is less than ν, whenever $i > N(\nu)$.

Proof: It follows from the definitions of N_1, N_2 and $f(\nu)$ and the analysis in this section. $\square$

It is worth to note that N is a logarithm depending on several parameters such as γ, σ, $N_\infty(n)$ and $\triangle_2 P'$ as well as an upper bound variable ν.

6.2. Sufficient subdivision iterations for homeomorphisms

For a Bézier curve $\mathcal{B}$ of degree 1 or 2, the control polygon is trivially simple and ambient isotopic to $\mathcal{B}$, provided the regularity of $\mathcal{B}$. So we consider $n \geq 3$.

Given any $\nu > 0$, Theorem 6.1 shows that there exists an integer $N(\nu)$, such that each exterior angle is less than ν after $N(\nu)$ subdivisions. Furthermore, there is an explicit closed formula to compute $N(\nu)$.

Theorem 6.2. *There exists a positive integer, $N(\frac{\pi}{n-1})$ for $n > 2$, where $N(\frac{\pi}{n-1})$ is defined by Equation 20, such that after $N(\frac{\pi}{n-1})$ subdivisions, each sub-control polygon will be simple.*

Proof: By Theorem 6.1, after $N(\frac{\pi}{n-1})$ subdivisions, each exterior angle is less than $\frac{\pi}{n-1}$. Since each sub-control polygon has a $n-1$ exterior angles, the total curvature of each sub-control polygon is less than π. Lemma 5.1 implies that this is a sufficient condition for each sub-control polygon being simple. □

While existence of sufficiently many iterations for the control polygon to fit inside the pipe $S_r(\mathcal{B})$ has been established, it remains of interest to bound the number of subdivisions that are sufficient for this containment. Define $N'(r)$ by

$$N'(r) = \frac{1}{2}\log(\frac{N_\infty(n)||\Delta_2 P||}{r}), \tag{21}$$

where r is the radius of a nonsingular pipe surface for $\mathcal{B}$. By the definition of $B_{dist}(i)$ (Equation 10) and Equation 21, we have $B_{dist}(i) < r$ whenever $i > N'(r)$.

Lemma 6.3. *The control polygon generated by i subdivisions, where $i > N'(r)$ and $N'(r)$ is given by Equation 21, satisfies*

$$\max_{t\in[0,1]} ||\mathcal{B}(t) - l(P,i)(t)|| < r,$$

and hence fits inside the pipe surface of radius r for $\mathcal{B}$.

Proof: By Lemma 6.1, $\max_{t\in[0,1]} ||\mathcal{B}(t) - l(P,i)(t)|| \leq B_{dist}(i)$. Then this lemma follows from the definition of $N'(r)$ given by Equation 21. □

While Theorem 6.2 addresses each sub-control polygon, it is of interest to ensure that the union of all these sub-control polygons is also simple. In Theorem 6.3, that union is the 'control polygon', as the result of multiple subdivisions.

Theorem 6.3. *Take any* $N \geq \max\{N(\frac{\pi}{2(n-1)}), N'(r)\}$*, where* $N(\nu)$ *is defined by Equations 20 and* $N'(r)$ *is given by Equation 21. Then after the* N*th subdivision, the control polygon will be simple.*

Proof: The inequality $N \geq N'(r)$ implies that the control polygon generated after the Nth subdivision lies inside the pipe. The inequality $N \geq N(\frac{\pi}{2(n-1)})$ ensures that the total curvature of its each sub-control polygon is less than $\frac{\pi}{2}$. These two conditions are sufficient conditions for the control polygon being simple (The proof of Theorem 5.2). $\square$

7. Conclusion

We proved the Angular Convergence and used it to show homeomorphism between a Bézier curve and its control polygon under subdivision. We showed that the rate of Angular Convergence is exponential. Plus the exponential rate of convergence in terms of Hausdorff distance, we established closed-form formulas to compute sufficient numbers of subdivision iterations to achieve topological properties.

The Angular Convergence is shown to be useful for determining homeomorphism. It is natural to consider ambient isotopy next [12], where total curvature will be fundamental, according to Fary-Milnor Theorem [16].

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